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Deficient functions of the random function *

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Abstract Two kinds of random functions are studied. It is proved that they have no deficient function almost surely.

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An interesting result in the theory of random functions is that random functions have no deficient values almost surely (a.s.). Because of some difficulties, there are only few discussions on small functions. In this paper, the authors concisely prove that two kinds of random functions have no deficient small functions a.s. The terminologies and notations are the same as in ref. [1], except those particularly explained.

Theorem 1[2]. *Suppose that* $f(z)$ *and* $\varphi_v(z)$ $(v=1,2,\cdots,q)$ *are meromorphic functions on the open plane,* $\varphi_v(z)$ *are distinct functions to each other, and*

$$T(r,\varphi_v) = o\{T(r,f)\},$$

then

$$\{q-1-o(1)\}T(r,f) < \sum_{v=1}^{q} N\left(r,\frac{1}{f-\varphi_v}\right) + q\bar{N}(r,f) + S(r,f),$$

where

$$S(r,f) = o\{\lg(rT(r,f))\},$$

except possibly some exceptional intervals of which the whole measure is finite.

Explanation. If $f(z)$ is not a constant, we may take $\{r_m\}\to\infty$, such that

$$\frac{S(r_m,f)}{T(r_m,f)}\to 0.$$

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Theorem 2. *Suppose* $f(z)$, $\varphi(z)$ *are non-constant analytic functions in* $|z|<\infty$ *and* $T(r,\varphi)=o\{T(r,f)\}$. *Let* $X(\omega)$ *be a continuous complex random variable on probability space* $(\Omega,\mathscr{A},P)$. *Then random function*

$$F_\omega(z)=f(z)+X(\omega)\varphi(z)$$

has no deficient function a.s. with the following meaning,

$$P\left\{\omega;\ \inf\left[\underline{\lim_{r\to\infty}}\frac{N\left(r,\dfrac{1}{F_\omega(z)-h(z)}\right)}{T(r,F_\omega)}\right]<1\right\}=0,$$

where take infimum for all complex numbers and all entire functions which satisfy $T(r,h)=o\{T(r,\varphi)\}$.

Proof. First we suppose that there are at most n distinct functions $F_\omega(z)$ that have deficient functions of which deficiency are greater than $\dfrac{1}{n}$ in Ω. Otherwise we have

$$F_j=f(z)+b_j\varphi(z)\quad j=1,2,\cdots,n+1,\tag{1}$$

where b_j are different from each other, thus F_j are different also. Each F_j have at least one deficient function

$$a_j(z);\ T(r,a_j)=o\{T(r,\varphi)\}\ (j=1,2,\cdots,n+1)$$

whose deficiency is greater than $\dfrac{1}{n}$, i.e.

$$1-\underline{\lim_{r\to\infty}}\frac{N\left(r,\dfrac{1}{F_j-a_j}\right)}{T(r,F_j)}>\frac{1}{n}.$$

Let $\varphi_j(z)=a_j(z)-b_j\varphi(z)$. By (1), $\varphi_j(z)$ are different from each other. By Theorem 1, we have

$$\{n+1-1+o(1)\}T(r,f)<\sum_{j=1}^{n+1}N\left(r,\frac{1}{f-\varphi_j}\right)+S(r,f)$$

$$=\sum_{j=1}^{n+1}N\left(r,\frac{1}{F_j-a_j}\right)+S(r,f).$$

By the above explanation, we take $\{r_m\}\to\infty$ and divide the above inequality by $T(r_m,f)$. Then let $m\to\infty$ and we have

$$n<(n+1)\left(1-\frac{1}{n}\right)=\frac{n^2-1}{n}.$$

This is a contradiction. Thus $F_\omega(z)$ that have deficient functions are at most countable in Ω. Since

$$F_{\omega_1}(z) = F_{\omega_2}(z) \Leftrightarrow X(\omega_1) = X(\omega_2),$$

the elements are also at most countable in the set

$$E = \{X(\omega) \in C;\ F_\omega(z) \text{ have deficient functions}\}.$$

Because $X(\omega)$ is a continuous complex random variable, we have

$$P\{\omega \in \Omega;\ X(\omega) \in E\} = 0.$$

Theorem 2 is proved.

Suppose that $\{(\Omega_j, \mathscr{A}_j, P_j)\}$ is an infinite sequence of probability space, and $(\Omega = \prod_{j=0}^{\infty} \Omega_j$, $\mathscr{A} = \prod_{j=0}^{\infty} \mathscr{A}_j$, $P = \prod_{j=0}^{\infty} P_j)$ a product probability space. Let $X(\omega) = (X_0(\omega_0), X_1(\omega_1), \cdots, X_j(\omega_j), \cdots)$ be a Rademacher random variable in the space $\Omega = \prod_{j=0}^{\infty} \Omega_j$, where the sequence $\{X_j(\omega_j)\}$ being independent and equally distributed satisfies $P_j(\omega_j;\ X_j(\omega_j) = 1) = P_j(\omega_j;\ X_j(\omega_j) = -1) = \frac{1}{2}$. Let complex number sequence $\{b_j\}$ satisfies

$$\overline{\lim_{n\to\infty}} \frac{n \ln n}{-\ln |b_n|} = \rho \in (0, \infty).$$

Then

$$f_\omega(z) = \sum_{n=0}^{\infty} X_n(\omega_n) b_n z^n \tag{2}$$

is a random function in Ω. Eq. (2) defines an entire function[3,4] of order ρ for $\omega = (\omega_0, \omega_1, \omega_2, \cdots) \in \Omega$ a.s.

Let

$$\Psi(\rho) = \left\{\psi = \sum_{n=0}^{\infty} \beta_n z^n;\ \overline{\lim_{n\to\infty}} \frac{n \ln n}{-\ln |\beta_n|} < \rho\right\}$$

be a set of small functions, i.e. $\Psi(\rho)$ is a set of all complex constants and all entire functions with order smaller than ρ.

Theorem 3. *The random series* (2) *has no small deficient function a.s. in product probability space* $(\Omega, \mathscr{A}, P)$, *i.e.*

$$P\left\{\omega;\ \inf\left(\overline{\lim_{r\to\infty}} \frac{N\left(r, \frac{1}{f_\omega - \psi}\right)}{T(r, f_\omega)};\ \psi \in \Psi(\rho)\right) < 1\right\} = 0 \quad \text{a.s.}$$

Proof. For any $\sigma \in \left(0, \frac{\rho}{3}\right)$, any $\Delta > 0$, let

$$\Psi(\rho - 3\sigma, \Delta)$$

$$= \left\{\psi = \sum_{n=0}^{\infty} \beta_n z^n \in \Psi(\rho - 3\sigma); \frac{n \ln n}{-\ln |\beta_n|} < \rho - 2\sigma, n > \Delta\right\}. \tag{3}$$

Take a sufficiently large positive integer p and let

$$E = E(p, \Delta, \sigma) = \{\omega; f_\omega(z) \text{ has deficient functions } \psi \in \Psi(\rho - 3\sigma, \Delta)$$
$$\text{of which deficiency is greater than } \frac{1}{p}\}.$$

Since

$$\{\omega; f_\omega(z) \text{ has deficient functions}\} = \bigcup_{\sigma > 0} \bigcup_{\Delta = 1}^{\infty} \bigcup_{p=1}^{\infty} E(p, \Delta, \sigma) \subset \Omega,$$

we need only to prove $P(E(p, \Delta, \sigma)) < \varepsilon$ for any $\varepsilon > 0$.

Take a monotonously increasing sequence $\{n(t)\}_{i=1}^{\infty}$, $n(1) > \Delta$ of positive integers, such that

$$\frac{n(t) \ln n(t)}{-\ln |b_{n(t)}|} > \rho - \varepsilon. \tag{4}$$

We suppose that $X_n(\omega_n) = \pm 1$ hold everywhere in Ω, otherwise delete a set of zero measures. Get

$$N > \frac{\log \frac{\varepsilon}{p+1}}{-\log 2}, \text{ i.e. } (p+1)\left(\frac{1}{2}\right)^N < \varepsilon.$$

For any $\omega'' = (\omega_{n(N)+1}, \omega_{n(N)+2}, \cdots) \in \Omega'' = \prod_{j=n(N)+1}^{\infty} \Omega_j$, there are at most $p + 1$ finite sequenes that consist of 1 and -1

$$A''(k; \omega'') = (a''_{n(1)}(k) = \pm 1, a''_{n(2)}(k) = \pm 1, \cdots, a''_{n(N)}(k) = \pm 1)$$
$$k = 1, 2, \cdots, u; \ u = u(\omega'') \leqslant p + 1, \tag{5}$$

and there are corresponding sequences

$$A'(k; \omega'') = \{a'_j(k) = \pm 1\} \quad k = 1, 2, \cdots, u,$$
$$j \in M = \{0, 1, 2, \cdots, n(N)\} - \{n(1), n(2), \cdots, n(N)\},$$

where j takes all over set M of non-negative integers that not greater than $n(N)$ except $\{n(t)\}_{t=1}^{N}$. Such that the corresponding u distinct functions

$$f_k = \sum_{n=n(N)+1}^{\infty} X_n(\omega_n) b_n z^n + \sum_{i=1}^{n(N)} a''_{n(t)}(k) b_{n(t)} z^{n(t)} + \sum_{n \in M}{}' a'_n(k) b_n z^n,$$
$$k = 1, 2, \cdots, u$$

have deficient functions $\psi_k^n = \sum_{n=0}^{\infty}\beta_n(k)z^n \in \Psi(\rho - 3\sigma, \Delta)$ whose deficiencies are greater than $\frac{1}{p}$ respectively, where $\sum'$ sums only the index n of non-negative integer that is not greater than $n(N)$ except $\{n(t)\}_{t=1}^{N}$. Otherwise $f = \sum_{n=n(N)+1}^{\infty} X_n(\omega_n)b_n z^n$ has at least $p+2$ distinct deficient functions

$$\psi_k = \psi_k^n - \sum_{t=1}^{n(N)} a''_{n(t)}(k)b_{n(t)}z^{n(t)} - \sum_{n\in M}{}' a'_n(k)b_n z^n, \quad (k = 1,2,\cdots,p+2).$$

By eq. (5), for any $i,j\in\{1,2,\cdots,p+2\}$, $A''(i;\omega'')\neq A''(j;\omega'')$, there is $v\in\{n(1), n(2), \cdots, n(N)\}$, such that $a''_v(i)\neq a''_v(j)$. By $|a''_v(i)| = |a''_v(j)| = 1$ and eqs. (3) and (4), we know that $\psi_i\neq\psi_j(i\neq j)$ and their deficiencies are greater than $\frac{1}{p}$. This contradicts Theorem 1.

Let the product space $\Omega' = \prod_{j=1}^{n(N)}\Omega_j$, $P' = \prod_{j=1}^{n(N)}P_j$. Note

$$\omega' = (\omega_0, \omega_1, \omega_2, \cdots, \omega_{n(N)}) \in \Omega',$$

$$D_k = D_k(\omega'') = \{\omega';\ X_{n(t)}(\omega_{n(t)}) = a''_{n(t)}(k),\ t = 1,\cdots,N)\},$$

$$D = D(\omega'') = \bigcup_{k=1}^{u} D_k(\omega'').$$

By independence, for any $\omega''\in\Omega''$, we have

$$P'(D) = \sum_{k=1}^{u}P'(D_k) = \sum_{k=1}^{u}\prod_{j=0}^{n(N)}P_j(X_j(\omega_j) = a''_j(k))$$

$$\leqslant \sum_{k=1}^{u}\prod_{t=1}^{N}P_{n(t)}(X_{n(t)}(\omega_{n(t)}) = a''_{n(t)}(k))$$

$$= u\left(\frac{1}{2}\right)^N \leqslant (p+1)\left(\frac{1}{2}\right)^N. \tag{6}$$

From the above, if $\omega = (\omega_0, \omega_1, \omega_2, \cdots) = (\omega', \omega'')\in E = E(p,\Delta,\sigma)\subset\Omega$, we have

$$\omega' = (\omega_0, \omega_1, \omega_2, \cdots, \omega_{n(N)}) \in D = D(\omega''). \tag{7}$$

By Fubini-Levl Theorem[5] and eqs. (6) and (7), we have

$$P(E) = E(1_E) = \lim_{m\to\infty}\int\cdots\iint 1_E P_0(d\omega_0)P_1(d\omega_1)\cdots P_m(d\omega_m)$$

$$= \lim_{m\to\infty}\int\cdots\int P_{n(N)+1}(d\omega_{n(N)+1})\cdots P_m(d\omega_m)\int\cdots\int 1_E P_0(d\omega_0)\cdots P_{n(N)}(d\omega_{n(N)})$$

$$\leqslant \lim_{m\to\infty}\int\cdots\int P_{n(N)+1}(d\omega_{n(N)+1})\cdots P_m(d\omega_m)\int\cdots\int 1_D P_0(d\omega_0)\cdots P_{n(N)}(d\omega_{n(N)})$$

$$\leqslant \lim_{m \to \infty} \int \cdots \int (p+1)\left(\frac{1}{2}\right)^N P_{n(N)+1}(d\omega_{n(N)+1}) \cdots P_m(d\omega_m) < \varepsilon .$$

The theorem is proved.

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